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CONTINUOUS AND INVERSE SHADOWING FOR FLOWS

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Abstract. We define continuous and inverse shadowing for flows and prove some properties. In particular, we will prove that an expansive flow without fixed points on a compact metric space which is a shadowing is also a continuous shadowing and hence an inverse shadowing (on a compact manifold without boundary).

1. Introduction. In the early 1990s it was realized that computer-simulated dynamical systems should have a property which would guarantee that resulting dynamics corresponds to the true one. Such a property was established and studied for discrete dynamical systems in a series of papers by Diamond and al. [**3, 4**] and it combined the classical shadowing with a property which was single out later in [**5**] and [**6**] and called inverse shadowing.

Generally speaking, inverse shadowing means that given a class of approximating methods, one can trace any (true) orbit with an arbitrary accuracy by an orbit generated with a precise enough method.

It has turned out that the inverse shadowing is a consequence of the shadowing if we assume some continuity condition which is satisfied under the assumption of hyperbolicity.

The aim of this paper is to introduce the concept of inverse shadowing for flows. As for some other notions, there are not straightforward relationships here between the discrete and continuous cases. Still, there are some similarities. In particular, we will prove that an expansive flow without fixed points on a compact metric space which is a shadowing is also a continuous shadowing, Theorem 3.6, and hence an inverse shadowing (on a compact manifold without boundary), Corollary 3.11.

2. Definitions and basic properties. We will denote by X a compact metric space with a distance d and by φ a flow on X (i.e $\varphi : X \times \mathbf{R} \to X$ is continuous and for all $x \in X$ and $s, t \in \mathbf{R}$ $\varphi(\varphi(x, s), t) = \varphi(x, s + t)$,

$\varphi(x,0) = x$). We shall write $\varphi_t x$ instead of $\varphi(x,t)$. The pair of sequences $\bar{x} = (\{x_i\}_{i=-\infty}^{\infty}, \{t_i\}_{i=-\infty}^{\infty})$ is said to be a (δ,a)–pseudo orbit ((δ,a,b)–pseudo orbit) of φ if $d(\varphi_{t_i} x_i, x_{i+1}) < \delta$ and $t_i \geq a > 0$ ($b \geq t_i \geq a > 0$) for all i.

For a given pseudo orbit we shall denote by $\bar{x} \star t$ the point

$$\bar{x} \star t = \begin{cases} \varphi_{(t - \sum_{i=0}^{i=n-1} t_i)} x_n & \text{when } \sum_{i=0}^{i=n-1} t_i \leq t < \sum_{i=0}^{i=n} t_i \quad \text{for } t \geq 0 \\ \varphi_{(t + \sum_{i=n}^{i=-1} t_i)} x_n & \text{when } -\sum_{i=n}^{i=-1} t_i \leq t < -\sum_{i=n+1}^{i=-1} t_i \quad \text{for } t < 0 \end{cases}$$

Here $\sum_m^n () = 0$ if $n < m$. It is easy to see that the function $\mathbf{R} \ni t \to \bar{x} \star t \in X$ is right continuous.

Two (δ,a) pseudo orbits $\bar{x}$ and $\bar{y}$ are equivalent ($\bar{x} \sim \bar{y}$) if $\bar{x} \star t = \bar{y} \star t$ for all $t \in \mathbf{R}$. Define

$$[\bar{x}] = \{\bar{y} : \bar{x} \sim \bar{y}\},$$

and then define $[\bar{x}] \star t = \bar{x} \star t$. We will also consider

$$Rep(\mathbf{R}) = \{\alpha : \mathbf{R} \to \mathbf{R} : \alpha \text{ is an increasing homeomorphism}, \quad \alpha(0) = 0\ \}$$

and

$$PO_a(\delta) = \{[\bar{x}] : \bar{x} \text{ is a } (\delta,a) \text{ pseudo orbit}\},$$
$$PO_a^b(\delta) = \{[\bar{x}] : \bar{x} \text{ is a } (\delta,a,b) \text{ pseudo orbit}\}.$$

We will often refer to an equivalence class as a pseudo orbit.

A pseudo orbit $[\bar{x}] \in PO_a(\delta)$ is ε–traced by a point $x \in X$ (or orbit of $x \in X$ is ε–traced by $[\bar{x}] \in PO_a(\delta)$) if there exists $\alpha \in Rep(\mathbf{R})$ such that we have $d(\varphi_{\alpha(t)} x, \bar{x} \star t) < \varepsilon$ for all $t \in \mathbf{R}$.

The flow has the *pseudo orbit tracing property* (POTP) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any $[\bar{x}] \in PO_a(\delta)$ is ε–traced by some $x \in X$. Thomas proved ([**8**]) that this definition does not depend on $a > 0$.

For $[\bar{x}], [\bar{y}] \in PO_a(\delta)$ we define the function

$$\rho([\bar{x}], [\bar{y}]) = \int_{\mathbf{R}} e^{-t^2} d(\bar{x} \star t, \bar{y} \star t) dt.$$

PROPOSITION 2.1. *For all $a, \delta > 0$ $(PO_a(\delta), \rho)$ is a metric space. Moreover, $\rho([\bar{x}_n], [\bar{y}]) \to 0$ $(n \to \infty)$ iff $\bar{x}_n \star t \to \bar{y} \star t$ $(n \to \infty)$ for all $t \in \mathbf{R}$.*

PROOF. Fix $a, \delta > 0$.

It's obvious that $\rho([\bar{x}], [\bar{y}]) = \rho([\bar{y}], [\bar{x}])$, $\rho([\bar{x}], [\bar{y}]) \leq \rho([\bar{x}], [\bar{z}]) + \rho([\bar{z}], [\bar{y}])$ and $\rho([\bar{x}], [\bar{x}]) = 0$ for all $[\bar{x}], [\bar{y}], [\bar{z}] \in PO_a(\delta)$.

If $\rho([\bar{x}], [\bar{y}]) = 0$ then $d(\bar{x} \star t, \bar{y} \star t) = 0$ almost everywhere. The function $\mathbf{R} \ni t \to d(\bar{x} \star t, \bar{y} \star t)$ is piecewise continuous and right continuous, so $\bar{x} \star t = \bar{y} \star t$ for all $t \in \mathbf{R}$.

Now, let us assume that $\bar{x}_n \star t \to \bar{y} \star t$ $(n \to \infty)$ for all $t \in \mathbf{R}$. Then $d(\bar{x} \star t, \bar{y} \star t) \to 0$ for all $t \in \mathbf{R}$. The Lebesgue Dominated Convergence Theorem implies that $\rho([\bar{x}_n], [\bar{y}]) \to 0$.

To prove the other implication assume that $\rho([\bar{x}_n],[\bar{y}]) \to 0$ $(n \to \infty)$. Now to show that $\bar{x}_n \star t \to \bar{y} \star t$ $(n \to \infty)$ for all $t \in \mathbf{R}$ assume this is not so. Then there exist $t_0 \in \mathbf{R}$, $\varepsilon_0 > 0$ and subsequence $k(n) \to \infty$ as $n \to \infty$ such that $d(\bar{x}_{k(n)} \star t_0, \bar{y} \star t_0) > \varepsilon_0$. Because of the compactness of X there exists $a > \beta > 0$ such that for all $x, y \in X$ if $d(x,y) > \varepsilon_0$, then $d(\varphi_t x, \varphi_t y) > \varepsilon_0/2$ for all $t \in [-\beta, \beta]$. There exist subsequence $k(s(n))$ and $0 < \lambda < \beta$ such that the functions $\bar{x}_{k(s(n))} \star t$ are continuous in $[t_0, t_0+\lambda]$ or in $[t_0-\lambda, t_0]$. Hence

$$\begin{aligned}\rho([\bar{x}_{k(s(n))}],[\bar{y}]) &\geq \int_{[t_0,t_0+\lambda]} e^{-t^2} d(\bar{x}_{k(s(n))} \star t, \bar{y} \star t)dt \\ &> \int_{[t_0,t_0+\lambda]} e^{-t^2} \varepsilon_0/2dt > M > 0\end{aligned}$$

or

$$\begin{aligned}\rho([\bar{x}_{k(s(n))}],[\bar{y}]) &\geq \int_{[t_0-\lambda,t_0]} e^{-t^2} d(\bar{x}_{k(s(n))} \star t, \bar{y} \star t)dt \\ &> \int_{[t_0-\lambda,t_0]} e^{-t^2} \varepsilon_0/2dt > N > 0,\end{aligned}$$

which is a contradiction completing the proof. □

Definition 2.2. A continuous map $\Phi : X \to PO_a(\delta)$ is a (δ, a)–method of φ if $\Phi(x) \star 0 = x$ for all $x \in X$.

Definition 2.3. The flow φ has the *continuous shadowing property* if for all $a > 0$ and $\varepsilon > 0$ there exist $\delta > 0$ and continuous map $W : PO_a(\delta) \to X$ such that every $[\bar{x}] \in PO_a(\delta)$ is ε–traced by $W([\bar{x}])$.

Definition 2.4. The flow φ has the *inverse shadowing property* (ISP) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any Φ being a $(\delta, 1)$–method of φ and any $y \in X$ there exists $x \in X$ such that $\Phi(x)$ is ε–traced by the orbit of y.

Now we prove that definition of the ISP does not depend on $a > 0$.

Proposition 2.5.

a) *The flow φ has the ISP iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that for every Φ being a $(\delta, 1, 2)$–method of φ and every $y \in X$ there exists $x \in X$ such that $\Phi(x)$ is ε–traced by the orbit of y.*
b) *For all $a > 0$, the flow φ has the ISP iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any Φ being a (δ, a)–method of φ and every $y \in X$ there exists $x \in X$ such that $\Phi(x)$ is ε–traced by the orbit of y.*
c) *For all $a > 0$, the flow φ has the ISP iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any Φ being a (δ, a, a)–method of φ and every $y \in X$ there exists $x \in X$ such that $\Phi(x)$ is ε–traced by the orbit of y.*

Proof. a) It is sufficient to prove that $PO_a(\delta) = PO_a^{2a}(\delta)$, for all $a > 0$. It is obvious that $PO_a^{2a}(\delta) \subset PO_a(\delta)$. Now take $[\bar{x}] \in PO_a(\delta)$. Then $\bar{x} = (\{x_i\}_{i=-\infty}^{\infty}, \{t_i\}_{i=-\infty}^{\infty})$, where $t_i \geq a$ for all $i \in \mathbf{Z}$. For each i there exist $m_i \in \mathbf{N}$ and $2a \geq r_i \geq a$ such that $t_i = m_i a + r_i$. Define

$$\bar{y} = (\{\ldots, \varphi_{m_{-1}a} x_{-1}, x_0, \varphi_a x_0, \ldots, \varphi_{m_0 a} x_0, x_1, \ldots\}, \\ \{\ldots, a, r_{-1}, a, \ldots, a, r_0, a, \ldots\})$$

We see that $[\bar{y}] \in PO_a^{2a}(\delta)$ and $\bar{x} \star t = \bar{y} \star t$ for all $t \in \mathbf{R}$. Hence $\bar{y} \in [\bar{x}]$.

b) Let assume that φ has the ISP and fix $a > 0$. There are two cases: $a \geq 1$ or $a < 1$. The first one is obvious, because $PO_a(\delta) \subset PO_1(\delta)$. So let's consider $a < 1$ and the smallest number $m \in \mathbf{N}$ such that $am > 1$.

For a given $\varepsilon > 0$ there exist $\delta_0, \delta_1, \ldots, \delta_m > 0$ satisfying the following conditions:

i) for any Ψ being a $(\delta_0, 1)$–method of φ and $y \in X$ there exists $x \in X$ such that $\Psi(x)$ $\varepsilon/2$–traces the orbit of y;

ii) $\delta_0 < \varepsilon/2$ and $d(x, y) < \delta_0$ implies $d(\varphi_t x, \varphi_t y) < \varepsilon/2$ for all $t \in [0, 1]$;

iii) $\delta_i < \delta_{i-1}/2$ and $d(x, y) < \delta_i$ implies $d(\varphi_t x, \varphi_t y) < \delta_{i-1}/2$ for $t \in [0, 1]$ for $i = 1, \ldots, m-1$.

Fix $\Phi : X \to PO_a(\delta_{m-1})$ method of φ and $y \in X$ and define

$$\Psi(x) = (\{\ldots, \Phi(x) \star (-1), x, \Phi(x) \star 1, \Phi(x) \star 2, \ldots\}, \{t_i = 1\}).$$

We prove that Ψ is a $(\delta_0, 1)$-method of φ and $d(\Phi(x) \star t, \Psi(x) \star t) < \varepsilon/2$ for all $x \in X$ and $t \in \mathbf{R}$. This will finish the proof, because by (i) there exists $x \in X$ and $\alpha \in Rep(\mathbf{R})$ such that $d(\varphi_{\alpha(t)} y, \Psi(x) \star t) < \varepsilon/2$. Then we will have

$$d(\varphi_{\alpha(t)} y, \Phi(x) \star t) \leq d(\varphi_{\alpha(t)} y, \Psi(x) \star t) + d(\Phi(x) \star t, \Psi(x) \star t) < \varepsilon.$$

Fix $k \in \mathbf{Z}$. There are at most m numbers $\lambda_1, \ldots, \lambda_m \in [0, 1]$ such that the function $[k, k+1] \ni t \to \Phi(x) \star t \in X$ is not continuous at the points $k + \lambda_i$ for $i = 1, \ldots, m$. By (iii) we have

$$d(\varphi_{\lambda_1}(\Phi(x) \star k), \Phi(x) \star (k + \lambda_1) < \delta_{m-1}$$

and

$$d(\varphi_{\lambda_2}(\Phi(x) \star k), \Phi(x) \star (k + \lambda_2) \\ < d(\varphi_{\lambda_2 - \lambda_1} \varphi_{\lambda_1}(\Phi(x) \star k), \Phi(x) \star (k + \lambda_1 + (\lambda_2 - \lambda_1))) \\ < \delta_{m-2}/2 + \delta_{m-1} < \delta_{m-2}.$$

After repeating this procedure $m - 2$ times we obtain

$$d(\varphi_1(\Phi(x) \star k), \Phi(x) \star (k + 1)) < \delta_0.$$

Hence $\Psi(x) \in PO_1(\delta_0)$ for all $x \in X$. By using this procedure and (ii) we have $d(\Phi(x) \star t, \Psi(x) \star t) < \varepsilon/2$ for all $x \in X$ and $t \in \mathbf{R}$.

Now we show that Ψ is continuous. Choose a sequence $x_n \to x$, as $n \to \infty$, and $t \in \mathbf{R}$. There exists $k \in \mathbf{Z}$ such that $t \in [k, k+1]$. Then

$$\Psi(x_n) \star t = \Psi(x_n \star (k + t - k) = \varphi_{t-k}(\Psi(x_n \star k)) = \varphi_{t-k}(\Phi(x_n \star k)).$$

By the continuoity of Φ , $\varphi_{t-k}(\Phi(x_n \star k)) \to \varphi_{t-k}(\Phi(x \star k)) = \Psi(x) \star t$. This completes the proof of (b).

c)

LEMMA 2.6. *For all $\varepsilon, \delta > 0$ there exist $\delta' > 0$ and a continuous map $P : PO_a^{2a}(\delta') \to PO_{2a}^{2a}(\delta)$, such that $d(\bar{x} \star t, P([\bar{x}]) \star t) < \varepsilon$ and $P([\bar{x}]) \star 0 = [\bar{x}] \star 0$, for all $t \in \mathbf{R}$ and $[\bar{x}] \in PO_a^{2a}(\delta')$.*

PROOF. The idea of this proof is the same as before. There exists $\delta' > 0$ such that $d(x, y) < \delta'$ implies $d(\varphi_t x, \varphi_t y) < \min(\varepsilon, \delta)$ for $t \in [0, 2a]$. Now we define

$$P([\bar{x}]) = (\{\ldots, [\bar{x}] \star (-2a), [\bar{x}] \star 0, [\bar{x}] \star 2a, [\bar{x}] \star 4a, \ldots\}, \{t_i = 2a\}).$$

Proof of the continuity of P follows as in (b). $\square$

Now let us assume that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any Φ being a (δ, a, a)–method of φ and every $y \in X$ there exists $x \in X$ such that $\Phi(x)$ is ε–traced by the orbit of y. We want to prove that then φ has the ISP with respect to $(\delta, a, 2a)$–methods. So fix $\varepsilon > 0$ and take $\delta > 0$ such that for any Φ being a (δ, a, a)–method of φ and every $y \in X$ there exists $x \in X$ such that $\Phi(x)$ is $\varepsilon/2$–traced by the orbit of y. Fix Φ, a $(\delta, a, 2a)$–method of φ, and take $\delta' > 0$ from Lemma 2.6 for $\varepsilon/2$. Let us consider $P \circ \Phi$, a (δ, a, a)–method of φ. Fix $y \in X$. There exists $x \in X$ such that $\Phi(x)$ $\varepsilon/2$–traces the orbit of x. Therefore $P(\Phi(x))$ ε–traces orbit of x.

This finishes the proof of Proposition 2.5. $\square$

Now we remind the definition of inverse shadowing property for homeomorphisms ([**2**]).

Let $f : X \to X$ be a homeomorphism. A sequence $\{x_i\}_{i \in \mathbf{Z}}$ is a *δ–pseudo orbit of f* if $d(f x_i, x_{i+1}) < \delta$. Let $PO(\delta)$ be a set of δ–pseudo orbits of f. A map $\Psi : X \ni x \to \{\Psi(x)_i\}_{i \in \mathbf{Z}} \in PO(\delta)$ is continuous if $x_n \to x$ $(n \to \infty)$ implies that for every $i \in \mathbf{Z}$ $\Psi(x_n)_i \to \Psi(x)_i$ $(n \to \infty)$. If $\Psi(x)_0 = x$ for any $x \in X$ then Ψ is called a (δ)–method of f. Finally we say that f has the inverse shadowing property if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every δ–method Ψ and $y \in X$ there exists $x \in X$ such that $\Psi(x)$ ε–traces the orbit of y (i.e. $d(\Psi(x)_i, f^i y) < \varepsilon$ for all $i \in \mathbf{Z}$).

For a given flow φ, we shall denote by φ_a a homeomorphism $\varphi_a : X \ni x \to \varphi_a x \in X$. Now we will prove the following:

PROPOSITION 2.7. *If there exists $a > 0$ such that φ_a has ISP then φ has the ISP.*

PROOF. In view of Proposition 2.5 it is sufficient to prove that φ has the ISP with respect to (a, a, δ)–methods.

Fix $\varepsilon > 0$. There is $\varepsilon' > 0$ such that for all $x, y \in X$ $d(x, y) < \varepsilon'$ implies $d(\varphi_t x, \varphi_t y) < \varepsilon$ for $t \in [0, a]$. Now take $\delta > 0$ from the definition of ISP for φ_a with respect to ε'. Fix $\Phi : X \to PO_a^a(\delta)$ a δ–method of f, and $y \in X$. Then $\Psi(x) = \{\Phi(x)\star(ia)\}_{i\in\mathbf{Z}}$ is a δ–method of φ_a and, moreover, $\Phi(x)\star(ia) = \Psi(x)_i$ for every $x \in X$ and $i \in \mathbf{Z}$. Since φ_a has the ISP, there exists $x \in X$ such that $d(\varphi_a^i y, \Psi(x)_i) < \varepsilon'$ for all $i \in \mathbf{Z}$. Hence

$$d(\varphi_{ia} y, \Phi(x) \star (ia)) < \varepsilon'$$

for $i \in \mathbf{Z}$. Now from the choice of ε' we have

$$d(\varphi_t y, \Phi(x) \star t) < \varepsilon$$

for $t \in \mathbf{R}$. This finishes the proof of Proposition 2.7. □

3. Continuous shadowing. In this section we prove that an expansive flow without fixed points on a compact metric space , which has POTP has the continuous shadowing property, and, as a corollary, that such flow on a compact manifold without boundary has the ISP.

The flow φ is *expansive* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(\varphi_t x, \varphi_{s(t)} y) < \delta$ for all $t \in \mathbf{R}$ for $x, y \in X$ and a continuous map $s : \mathbf{R} \to \mathbf{R}$ with $s(0) = 0$, then $y = \varphi_t x$, where $|t| < \varepsilon$.

DEFINITION 3.1. The flow φ has *pseudo orbit tracing property* (POTP) if for all $\varepsilon > 0$ there exists $\delta > 0$ such that each $[\bar{x}] \in PO_1(\delta)$ is ε–traced by an orbit of φ.

Thomas proved in [**8**] that φ has the POTP iff for all $a > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that each $[\bar{x}] \in PO_a(\delta)$ is ε–traced by an orbit of φ. We recall the following

LEMMA 3.2. ([**1**]) *Let φ be a continuous flow on X without fixed points. Then φ is expansive if and only if for all $\varepsilon > 0$ there exsists $r > 0$ such that if $t = (t_i)_{i\in\mathbf{Z}}$ and $u = (u_i)_{i\in\mathbf{Z}}$ are doubly infinite sequences of real numbers with $t_0 = u_0 = 0$, $0 < t_{i+1} - t_i \le r$, $|u_{i+1} - u_i| < r$, $t_i \to \infty$, $t_{-i} \to -\infty$, as $i \to \infty$, and if $x, y \in X$ satisfy $d(\varphi_{t_i} x, \varphi_{u_i} x) \le r$ for all $i \in \mathbf{Z}$, then there exists t such that $|t| < \varepsilon$ and $y = \varphi_t x$.*

LEMMA 3.3. ([**1**]) *If a flow φ has no fixed points, then there is $T_0 > 0$ such that for every T satisfying $0 < T < T_0$ there exists $\gamma > 0$ with $d(\varphi_T x, x) \ge \gamma$ for all $x \in X$.*

Lemma 3.4. ([**8**]) *Let $(\alpha_i)_{i\in\mathbf{N}}$ be a family of continuous increasing functions from $[0,b]$ into $\mathbf{R}$ with $\alpha_i(0)=0$ for all i, and assume $\alpha_i(b)\to\infty$ as $i\to\infty$. Then for every $\lambda,\beta>0$ there are j and $t_1,t_2\in[0,b]$ such that $0<t_2-t_1<\lambda$ and $\alpha_j(t_2)-\alpha_j(t_1)=\beta$, where $[0,b]$ is a closed interval in $\mathbf{R}$.*

Lemma 3.5.

a) *For all $a,\delta,\varepsilon>0$ there is $\lambda>0$ such that for all $[\bar{x}]\in PO_a(\delta)$ and $t_1,t_2\in\mathbf{R}$, $|t_1-t_2|<\lambda$ implies $d([\bar{x}]\star t_1,[\bar{x}]\star t_2)<\delta+\varepsilon$;*
b) *For all $a,\delta>0$ and $([\bar{y}_n])_{n\in\mathbf{N}}$, a sequence in $PO_a^a(\delta)$, such that $[\bar{y}_n]\to[\bar{y}]\in PO_a^a(\delta)$ and for all $\eta,T>0$, there exists a sequence $(k(n))_{n\in\mathbf{N}}$ in $\mathbf{N}$ such that $d([\bar{y}_{k(n)}]\star t,[\bar{y}]\star t)<\eta$ for all $t\in[-T,T]$ and $n\in\mathbf{N}$;*
c) *Let $[\bar{x}]\in PO_a^a(\delta)$. Then for any $\eta,T>0$ there exist U a neighborhood of $[\bar{x}]$ such that for every $[\bar{x}]'\in U$ and $t\in[-T,T]$ $d([\bar{x}]\star t,[\bar{x}]'\star t)<\eta$.*

Proof. a) For a given $\varepsilon>0$ there is $\lambda'>0$ such that $d(\varphi_t x,x)<\varepsilon/2$ for all $x\in X$ and $t\in[-\lambda',\lambda']$. Now take $\lambda=\min\{\lambda',a\}$ and fix $t_1,t_2\in\mathbf{R}$ such that $|t_1-t_2|<\lambda$. There is only one $t_0\in[t_1,t_2]$ such that the function $[\bar{x}]*t$ is not continuous at $t=t_0$. Hence $d([\bar{x}]\star t_1,[\bar{x}]\star t_2)\le d([\bar{x}]\star t_1,[\bar{x}]\star t_0)+d([\bar{x}]\star t_0,[\bar{x}]\star t_2)<\varepsilon+\delta$.

b) Let assume that it is not so. Then there exist $a,\delta,\eta,T>0$ and sequences $([\bar{y}_n])_{n\in\mathbf{N}}\subset PO_a^a(\delta)$, $(t_n)_{n\in\mathbf{N}}\subset[-T,T]$ and $(k(n))_{n\in\mathbf{N}}\subset\mathbf{N}$ such that $[\bar{y}_n]\to[\bar{y}]\in PO_a^a(\delta)$,$t_n\to t_0\in[-T,T]$ and $d([\bar{y}_{k(n)}]\star t_n,[\bar{y}]\star t_n)\ge\eta$.

We can assume that there exist $k\in\mathbf{Z}$ such that $t_n\in[ka,(k+1)a]$ for all $n\in\mathbf{N}$. Then $[\bar{y}_{k(n)}]\star t_n=\varphi_{t_n-ka}([\bar{y}_{k(n)}]\star(ka))$ for all $n\in\mathbf{N}$. Therefore

$$d(\varphi_{t_0-ka}([\bar{y}_{k(n)}]\star(ka)),\varphi_{t_0-ka}([\bar{y}]\star(ka)))\ge\eta.$$

It is a contradiction, because $[\bar{y}_{k(n)}]\star(ka)\to[\bar{y}]\star(ka)$.

c) Proof is very similar to the above proof of (b).

This finishes the proof of Lemma 3.5. □

The idea of the proof of Theorem 3.6 to some extent follows the proof of Theorem 3 in [**8**], stating that the expansiveness and POTP imply topological stability.

Theorem 3.6. *For every expansive flow φ on X without fixed points, if it has the POTP, then it has the continuous shadowing property.*

Proof. Firstly we prove that for all $a>0$ and $\varepsilon>0$ there exist $\delta>0$ and continuous map $W':PO_a^a(\delta)\to X$ such that every $[\bar{x}]\in PO_\delta$ is ε–traced by $W'([\bar{x}])$. (Then we will use Lemma 2.6).

Let us fix $a>0$. Suppose $\varepsilon>0$ is given. We can assume that $\varepsilon<T_0/2$ (with T_0 as in Lemma 3.3). Using Lemma 3.2 take $0<r<\varepsilon$. By Lemma 3.4 there is $\gamma>0$ such that $d(\varphi_{(r/2)}x,x)\ge\gamma$ for all $x\in X$. In the definition of expansiveness, take $\varepsilon'<\gamma$, $\varepsilon'<r$ such that if $d(\varphi_{s(t)}x,\varphi_t y)\le\varepsilon'$ for $x,y\in X$

and a continuous map with $s(0) = 0$, then $y = \varphi_t x$, where $|t| \le r$. Also choose $\varepsilon'/24 > \delta > 0$ such that every $[\bar{x}] \in PO_a^a(\delta)$ is $\varepsilon'/6$–traced by an orbit of φ.

For a given $[\bar{x}] \in PO_a^a(\delta)$, there are $z \in X$ and $\alpha \in Rep(\mathbf{R})$ such that $d([\bar{x}] * t, \varphi_{\alpha(t)} z) < \varepsilon'/6$. Assume that $z' \in X$ also $\varepsilon'/6$–traces $[\bar{x}]$. Then $d(\varphi_{\alpha(t)} z, \varphi_{\alpha'(t)} z') < \varepsilon'/3$ for all $t \in \mathbf{R}$. By the expansiveness and our choice of ε' imply that $z' = \varphi_t z$ with $|t| < \varepsilon$. Hence every $[\bar{x}] \in PO_a^a(\delta)$ is uniquely traced by an orbit of φ, say $(\varphi_t z)_{t \in \mathbf{R}}$. Now for $[\bar{x}]$ define:

$$A_{\bar{x}} = \{x \in X : \forall_{\eta, T > 0}\ \exists_{\alpha \in Rep(\mathbf{R})}\ \forall_{t \in [-T,T]}\ \ d([\bar{x}] * t, \varphi_{\alpha(t)} x) < \varepsilon'/6 + \eta\}.$$

This set is not empty, because $z \in A_{\bar{x}}$.

Lemma 3.7.

a) $A_{\bar{x}} \subset (\varphi_t z)_{t \in \mathbf{R}}$ *and the time diameter of* $A_{[\bar{x}]}$ *is less than* ε*;*
b) $A_{\bar{x}}$ *is closed in* X.

Proof. a) Fix $x \in A_{\bar{x}}$. We will show that $x = \varphi_t z$ with some $|t| < \varepsilon$.

Let $(\eta_i)_{i \in \mathbf{N}}$ and $(T_i)_{i \in \mathbf{R}}$ be sequences of positive real numbers such that $\eta_i < \varepsilon'/6$, $\eta_i \to 0$ and $T_i \to \infty$ as $i \to \infty$. We know that $x \in A_{\bar{x}}$, so there is $\alpha_i \in Rep(\mathbf{R})$ such that

$$d([\bar{x}] * t, \varphi_{\alpha_i(t)} x) < \varepsilon'/6 + \eta_i \quad \text{for all} \quad t \in [-T_i, T_i].$$

Using the fact that $[\bar{x}]$ is $\varepsilon'/6$–traced by z with $\alpha \in Rep(\mathbf{R})$, we get

$$d(\varphi_{\alpha(t)} z, \varphi_{\alpha_i(t)} x) < \varepsilon'/3 + \eta_i \quad \text{for all} \quad t \in [-T_i, T_i].$$

Let $T_i' = \min\{|\alpha(T_i)|, |\alpha(-T_i)|\}$ and $\gamma_i = \alpha_i \circ \alpha^{-1}$. It is clear that $T_i' \to \infty$ and $\gamma_i \in Rep(\mathbf{R})$. Now we have

$$d(\varphi_u z, \varphi_{\gamma_i(u)} x) < \varepsilon'/3 + \eta_i \quad \text{for all} \quad u \in [-T_i', T_i'].$$

By the continuity of γ_i we choose $0 < s_i < r$ such that $|u - u'| < s_i$ implies $|\gamma_i(u) - \gamma_i(u')| < r/2$. Now

$$\begin{aligned} d(\varphi_{\gamma_{i+1}(u) - \gamma_i(u)} \varphi_{\gamma_i(u)} x, \varphi_{\gamma_i(u)} x) &= d(\varphi_{\gamma_{i+1}(u)} x, \varphi_{\gamma_i(u)} x) \\ &\le d(\varphi_{\gamma_{i+1}(u)} x, \varphi_u z) + d(\varphi_u z, \varphi_{\gamma_i(u)} x) \\ &< \frac{2}{3}\varepsilon' + 2\eta_i < \varepsilon', \end{aligned}$$

for all $u \in [-T_i', T_i']$.

Since γ_i, γ_{i+1} are continuous, $\gamma_i(0) = \gamma_{i+1}(0) = 0$ and $\varepsilon' < \gamma$, it follows that $|\gamma_{i+1}(u) - \gamma_i(u)| < r/2$. Hence $|u - u'| < s_{i+1}$ implies $|\gamma_{i+1}(u') - \gamma_i(u)| < r$.

Fix $i \in \mathbf{Z}$. We can choose a strictly increasing sequence of real numbers $(t_j)_{j \in \mathbf{Z}}$ with $t_0 = 0$ such that if $u_j \in [0, T_i']$, then $t_{j+1} - t_j < s_{i+1}$ for $j > 0$. If $u_j \in [-T_i', 0]$, then $t_{j+1} - t_j < s_{i+1}$ for $j < 0$. For $t_j \in [-T_i', T_i']$, let $u_j = \gamma_i(t_j)$ and for $t_j \in [-T_{i+1}', -T_i') \cup (T_i', T_{i+1}']$ let $u_j = \gamma_{i+1}(t_j)$. So $|u_{j+1} - u_j| < r$ and

$d(\varphi_{u_j}x, \varphi_{t_j}z) < \varepsilon' < r$. Using Lemma 3.2 we have $x = \varphi_t z$ with $|t| < \varepsilon$, which proves (a).

b) Because of (a) we only need to show that $A_{\bar{x}}$ is closed in the orbit $(\varphi_t z)_{t\in\mathbf{R}}$ with respect to the relative topology. Let $(z_i)_{i\in\mathbf{N}} \subset A_{\bar{x}}$ be a sequence such that $z_i \to z' \in (\varphi_t z)_{t\in\mathbf{R}}$. We will show that $z' \in A_{\bar{x}}$. So fix $\eta, T > 0$. There are $\alpha_i \in Rep(\mathbf{R})$ such that for all i and $t \in [-T, T]$

$$d([\bar{x}] * t, \varphi_{\alpha_i(t)}z_i) < \varepsilon'/6 + \eta/2.$$

Since all z_i and z' are in the same orbit with time distance ε, there exists (because of the expansiveness) an integer N large enough such that for $i > N$ and $t \in \mathbf{R}$

$$d(\varphi_t z_i, \varphi_t z') < \eta/2.$$

Hence for $t \in [-T, T]$

$$d([\bar{x}] * t, \varphi_{\alpha_i(t)}z') < \varepsilon'/6 + \eta/2 + \eta/2 = \varepsilon'/6 + \eta.$$

It follows that $z' \in A_{\bar{x}}$, which finishes the proof of Lemma 3.7. □

A point $x \in A_{\bar{x}}$ is called *the largest limit* ($L.A_{\bar{x}}$) of $A_{\bar{x}}$ if $x = \varphi_t x'$ with $t \geq 0$ for all $x' \in A_{\bar{x}}$. Such point is unique. Now we define $W' \ni [\bar{x}] \to L.A_{\bar{x}} \in X$. We will prove that for each compact $K \subset PO^a_a(\delta)$ a map $W'|_K$ is continuous. Fix such a K.

For $\eta, T > 0$, let us define the following set

$$A_{\bar{x},\eta,T} = \{x \in X : \exists_{\alpha\in Rep(\mathbf{R})}\ \forall_{t\in[-T,T]}\ \ d([\bar{x}] * t, \varphi_{\alpha(t)}x) < \varepsilon'/6 + \eta\}.$$

Lemma 3.8. *For all $\lambda > 0$ and $[\bar{x}] \in PO^a_a(\delta)$ there exist $\eta, T > 0$ such that $d(x, A_{\bar{x}}) < \lambda$ for all $x \in A_{\bar{x},\eta,T}$.*

Proof. Let $(\eta_i)_{i\in\mathbf{N}}$ and $(T_i)_{i\in\mathbf{R}}$ be sequences of positive real numbers such that $\eta_i < \varepsilon'/6$, $\eta_i \to 0$ and $T_i \to \infty$ as $i \to \infty$. Also assume that $z_i \in A_{\bar{x},\eta_i,T_i}$ and $z_i \to z$. We will show that then $z \in A_{\bar{x}}$(i).

There are $\alpha_i \in Rep(\mathbf{R})$ such that, for $i \in \mathbf{N}$ and $t \in [-T_i, T_i]$,

$$d(\varphi_{\alpha_i(t)}z_i, [\bar{x}] * t) < \varepsilon'/6 + \eta_i. \tag{1}$$

Since $z_i \to z$, we can take $(\beta_i)_{i\in\mathbf{N}}$ and $(w_i)_{i\in\mathbf{R}}$ sequences of positive real numbers with $\beta_i < \varepsilon'/6$, $\beta_i \to 0$ and $w_i \to \infty$ as $i \to \infty$, such that $d(\varphi_t z_i, \varphi_t z) < \beta_i$ for all $t \in [-w_i, w_i]$. Thus for all $t \in [-\alpha_i^{-1}(w_i), \alpha_i^{-1}(w_i)]$ we have

$$d(\varphi_{\alpha_i(t)}z_i, \varphi_{\alpha_i(t)}z) < \beta_i. \tag{2}$$

Suppose that we have already known that

$$v_i = \min\{|\alpha_i^{-1}(w_i)|, |\alpha_i^{-1}(-w_i)|\} \to \infty. \tag{$*$}$$

Then, because of (1) and (2) we have

$$\begin{aligned} d(\varphi_{\alpha_i(t)}z, [\bar{x}] * t) &\leq d(\varphi_{\alpha_i(t)}z, \varphi_{\alpha_i(t)}z_i) + d(\varphi_{\alpha_i(t)}z_i, [\bar{x}] * t) \\ &< \beta_i + \varepsilon'/6 + \eta_i. \end{aligned}$$

for $t \in [-k_i, k_i]$, where $k_i = \min\{v_i, T_i\}$. So $z \in A_{\bar{x}, \eta_i+\beta_i, k_i}$. Hence $z \in A_{\bar{x}}$.

Now we prove $(*)$. Firstly we show that $\alpha_i^{-1}(w_i) \to \infty$. Assume this is not so. There are $b > 0$ and $N \in \mathbf{N}$ such that $\alpha_i^{-1}(w_i) \leq b$ for $i > N$. Then $w_i \leq \alpha_i(b)$, so $\alpha_i(b) \to \infty$ as $i \to \infty$.

Using Lemma 3.4 and Lemma 3.5(a), we may find $j \in \mathbf{N}$ and $t_1, t_2 \in [0, b]$ such that $d([\bar{x}] \star t_1, [\bar{x}] \star t_2) < \delta + \delta < \varepsilon'/12$ and $\alpha_j(t_2) - \alpha_j(t_1) = r/2$. Hence $d(\varphi_{\alpha_j(t_1)}z_j, \varphi_{\alpha_j(t_2)}z_j) \geq \gamma > \varepsilon'$.

But there is

$$\begin{aligned} d(\varphi_{\alpha_j(t_1)}z_j, \varphi_{\alpha_j(t_2)}z_j) &\leq d(\varphi_{\alpha_j(t_1)}z_j, [\bar{x}] \star t_1) + d([\bar{x}] \star t_1, [\bar{x}] \star t_2) \\ &+ d([\bar{x}] \star t_2, \varphi_{\alpha_j(t_2)}z_j) < \varepsilon'. \end{aligned}$$

This is a contradiction. We may similarly prove that $\alpha_i^{-1}(w_i) \to \infty$.

Now if we assume that $d(z_i, A_{\bar{x}}) \geq \lambda$ for all $i \in \mathbf{N}$, then $d(z, A_{\bar{x}}) \geq \lambda$. This is a contradiction, which finishes the proof of Lemma 3.8. □

LEMMA 3.9. *For all $\lambda > 0$ there exist $\eta, T > 0$ such that for all $x \in A_{\bar{x},\eta,T}$ and for all $[\bar{x}] \in K$ $d(x, A_{\bar{x}}) < \lambda$.*

PROOF. This is a consequence of Lemma 3.8 and the compactness of K.

Indeed, take $[\bar{x}] \in K$. By Lemma 3.8 there exist $\eta_{\bar{x}}, T_{\bar{x}} > 0$ such that $d(x, A_{\bar{x}}) < \lambda/2$ for $x \in A_{\bar{x},\eta_{\bar{x}},T_{\bar{x}}}$. Now, by Lemma 3.5 (c) there exists $U_{\bar{x}}$, a neighborhood of $[\bar{x}]$, such that $d([\bar{x}] * t, [\bar{x}]' \star t) < \eta_{\bar{x}}/2$ for $[\bar{x}]' \in U_{\bar{x}}$ and $t \in [-T_{\bar{x}}, T_{\bar{x}}]$. So $A_{\bar{x}',\eta_{\bar{x}}/2,T_{\bar{x}}} \subset A_{\bar{x},\eta_{\bar{x}},T_{\bar{x}}}$ for all $[\bar{x}]' \in U_{\bar{x}}$.

By the compactness of K there are $[\bar{x}]_1, \ldots, [\bar{x}]_k$ with an open cover $U_{\bar{x}_1}, \ldots, U_{\bar{x}_k}$. Let $\eta = \frac{1}{2}\min_{1\leq i\leq k}\{\eta_{\bar{x}_k}\}$ and $T = \max_{1\leq i\leq k}\{T_{\bar{x}_k}\}$.

Now take $x \in A_{\bar{x},\eta,T}$. There exists $1 \leq j \leq k$ such that $A_{\bar{x},\eta,T} \subset A_{\bar{x}_j,\eta_j,T_j}$, thus $d(A_{\bar{x}}, A_{\bar{x}_j}) < \lambda/2$. Finally we have

$$d(x, A_{\bar{x}}) \leq d(x, A_{\bar{x}_j}) + d(A_{\bar{x}_j}, A_{\bar{x}}) < \lambda.$$

This finishes the proof of Lemma 3.9. □

LEMMA 3.10. *Let $([\bar{x}_i])_{i\in\mathbf{N}}$ be a sequence in K such that $[\bar{x}_i] \to [\bar{x}]$. If $z_i \in A_{\bar{x}_i}$ and $z_i \to z$, then $z \in A_{\bar{x}}$.*

PROOF. Let $\{\lambda_i\}$ be a sequence of positive real numbers such that $\lambda_i \to 0$. By Lemma 3.9 there are $\eta_i, T_i > 0$ such that $d(x, A_{\bar{x}}) < \lambda_i$ for $x \in A_{\bar{x},\eta_i,T_i}$. Using Lemma 3.5 (b) we see that there is subsequence $[\bar{x}_{j_i}]$ of $[\bar{x}_i]$ which satisfies

$$d([\bar{x}_i] \star t, [\bar{x}_{j_i}] \star t) < \eta_i/2$$

for $t \in [-T_i, T_i]$ and $i \in \mathbf{N}$.

But $z_{j_i} \in A_{\bar{x_{j_i}}}$, so there are $\alpha_i \in Rep(\mathbf{R})$ such that

$$d(\varphi_{\alpha_i(t)} z_{j_i}, [\bar{x}_{j_i}] \star t) < \varepsilon'/6 + \eta_i/2$$

for $t \in [-T_i, T_i]$. Therefore

$$d(\varphi_{\alpha_i(t)} z_i, [\bar{x}_{j_i}] \star t) \leq d(\varphi_{\alpha_i(t)} z_i, \varphi_{\alpha_i(t)} z_{j_i}) + d(\varphi_{\alpha_i(t)} z_{j_i}, [\bar{x}_{j_i}] \star t) < \varepsilon'/6 + \eta_i.$$

This means that $z_{j_i} \in A_{\bar{x},\eta_i,T_i}$. We have $d(z_{j_i}, A_{\bar{x}}) < \lambda_i$, so $d(z, A_{\bar{x}}) = 0$. Since $A_{\bar{x}}$ is closed, $z \in A_{\bar{x}}$. This finishes the proof of Lemma 3.10. □

Now we shall show that $W' : K \to X$ is continuous (where K is a compact subset of $PO_a^a(\delta)$).

Let assume that sequences $\{[\bar{x}_i]\}$,$\{z_i\}$ are such that $z_i = L.A_{\bar{x}_i}$ and $[\bar{x}_i] \to [\bar{x}]$. Let $z = L.A_{\bar{x}}$. We want to show that $z_i \to z$. Owing to the compactness of X and Lemma 3.10 we may assume that $z_i \to z' \in A_{\bar{x}}$. Let x be any point in $A_{\bar{x}}$ and let $\{\lambda_i\}$ be a sequence of positive real numbers such that $\lambda_i \to 0$. Choose $\eta_i, T_i > 0$ as in Lemma 3.9. Now there exists a subsequence $\{[\bar{x}_{j_i}]\}$ such that

$$d([\bar{x}_i] \star t, [\bar{x}_{j_i}] \star t) < \eta_i/2$$

for $t \in [-T_i, T_i]$ and $i \in \mathbf{N}$. Since $x \in A_{\bar{x}}$, there exists $\alpha_i \in Rep(\mathbf{R})$ such that

$$d(\varphi_{\alpha_i(t)} x, [\bar{x}] * t) < \varepsilon'/6 + \eta_i/2$$

for $t \in [-T_i, T_i]$. So we have

$$d(\varphi_{\alpha_i(t)} x, [\bar{x}_{j_i}] \star t) < \varepsilon'/6 + \eta_i$$

for $t \in [-T_i, T_i]$ and $i \in \mathbf{N}$. Therefore, $x \in A_{\bar{x}_{j_i},\eta_i,T_i}$. Lemma 3.9 implies that $d(x, A_{\bar{x}_{j_i}}) < \lambda_i$. Since $A_{\bar{x}_{j_i}}$ is closed, we can choose $x_{j_i} \in A_{\bar{x}_{j_i}}$ such that $d(x, x_{j_i}) = d(x, A_{\bar{x}_{j_i}}) = \lambda_i$. Then $x_{j_i} \to x$. Since $z_{j_i} = L.A_{\bar{x_{j_i}}}$ there are $w_{j_i} \geq 0$ such that $z_{j_i} = \varphi_{w_{j_i}} x_{j_i}$. Hence $z' = \varphi_w x$ with $w \geq 0$ for every $x \in A_{\bar{x}}$. So $z' = L.A_{\bar{x}}$. By uniqueness of $L.A_{\bar{x}}$, $z = z'$. This proves that every convergent subsequence of z_i has z as a limit. It means that $z_i \to z$.

We have proved that $W'|K$ is continuous for every compact $K \subset PO_a^a(\delta)$. So $W' : PO_a^a(\delta) \to X$ is continuous because $PO_a^a(\delta)$ is a metric space.

Now we will prove that φ has the continuous shadowing property. Fix $a, \varepsilon > 0$. There exist $\delta > 0$ and continuous map $W' : PO_{2a}^{2a}(\delta) \to X$ such that every $[\bar{x}] \in PO_{2a}^{2a}(\delta)$ is $\varepsilon/2$–traced by $W'([\bar{x}])$. Using Lemma 2.6 we have $\delta' > 0$ and a continuous map $P : PO_a^{2a}(\delta') \to PO_{2a}^{2a}(\delta)$ such that $d(P([\bar{x}]) \star t, [\bar{x}] * t) < \varepsilon/2$. Now take $W = W' \circ P$. It finishes proof of Theorem 3.6, because $PO_a^{2a}(\delta') = PO_a(\delta')$. □

Corollary 3.11. *Every expansive flow φ on a compact manifold M without boundary, without fixed points, and with the POTP, has the inverse shadowing property.*

Proof. There exists $\varepsilon' > 0$ such that a continuous map $H : M \to M$ with $d(H, id) < \varepsilon'$ is suriective.

Fix $0 < \varepsilon < \varepsilon'$ and $\delta > 0$ such that we have a continuous map $W : PO_1(\delta) \to M$ from Theorem 3.6. Now fix a method $\Phi : M \to PO_1(\delta)$ and $y \in M$. Let us define $H = W \circ \Phi$. For all $x \in M$, there is

$$d(H(x), x) = d(W(\Phi(x)), \Phi(x) \star 0) < \varepsilon,$$

because $W(\Phi(x))$ ε–traces $\Phi(x)$. Hence H is suriective. So there is $x \in M$ such that $H(x) = y$. Now it is obvious that $\Phi(x)$ is ε–traced by the orbit of y. This finishes the proof of Corollary 3.11. □

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